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The Hausdorff Dimension of Some Fractals and Attractors of Overlapping Construction

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We introduce a method to show that the Hausdorff dimension of certain fractals of overlapping construction is "almost always" what would be observed if no overlapping occurred. The method is also used to examine the dimension of attractors in some noninjective, piecewise linear, "baker's"-type transformations.

KEY WORDS: Hausdorff dimension; fractal; attractor.

1. INTRODUCTION

The standard "Cantor" method of construction of self-similar fractals in n-dimensional space consists of replacing a set E by k similar subsets $E_1,..., E_k$ with similarity ratios $\lambda_1,..., \lambda_k$, repeating this process for each of the E_i, and so on. Many familiar examples, such as the von Koch curve or the Menger sponge, may be obtained in this way. Provided that the components in each stage of the construction do not overlap "too much," the Hausdorff dimension of the fractal is given by the unique positive s satisfying

$$\sum_1^k \lambda_i^s = 1 \tag{1}$$

If the component sets are allowed to overlap substantially, a limiting set is still obtained, but the problem of calculating the dimension becomes much more awkward. Our first aim in this paper is to show that "in general" the dimension will still be given by (1) despite the overlapping.

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A comparable situation arises in discrete dynamical systems. If T is a smooth, injective transformation, the attractors that can occur, such as the "horseshoe" attractor, have been widely analyzed. Suppose now that T is not injective (for example, imagine a horseshoe mapping with the ends of the horseshoe bent to cross over). The attractors and ω-limit sets of the transformation may still exist, but their structure can be rather more complex due to overlapping of various parts. Nevertheless, it seems plausible that in the "generic" case the dimension of the attractor is what would be expected for a transformation with similar local characteristics but with no overlapping. As a preliminary model of this phenomenon, we examine a piecewise linear "slanting baker's transformation" and show that the dimension of the attractor is almost surely what would be expected were no overlapping present.

2. THE DIMENSION OF FRACTALS INVARIANT UNDER SIMILARITIES

Self-similar subsets of $\mathbb{R}^n$ may be conveniently regarded as the invariant sets of certain transformations. Let $S_1,\ldots, S_k$ be similarity transformations, so that $|S_i x - S_i y| = \lambda_i |x-y|$ $(x, y \in \mathbb{R}^n)$, where λ_i is the contraction ratio of S_i. An application of the contraction mapping theorem with the Hausdorff metric shows that there exists a unique, nonempty compact $F \subset \mathbb{R}^n$ such that

$$F = \bigcup_{i=1}^{k} S_i(F) \tag{2}$$

(see Hutchinson[4]). If I is any compact set with $S_i(I) \subset I$ for $1 \leqslant i \leqslant k$, then $F = \bigcap_{r=1}^{\infty} F_r$, where $F_r = \bigcup S_{i_1} \circ \cdots \circ S_{i_r}(I)$, with the union over all sequences with $1 \leqslant i_j \leqslant k$ for each j. We say that the *open set condition* holds if there exists a bounded open set J such that

$$J \supset \bigcup_{i=1}^{k} S_i(J) \tag{3}$$

with the union disjoint; thus, taking $I = \bar{J}$ in the construction of F above, this means that the components of each F_r cannot overlap very much. Hutchinson (see also Moran[8]) shows that if the open set condition holds for a set of similarities $S_1,\ldots, S_k$, then $\dim F$, the Hausdorff dimension of F, equals the unique value of s for which

$$\sum_{i=1}^{k} \lambda_i^s = 1 \tag{4}$$

For example, taking $S_1, S_2: \mathbb{R} \circlearrowleft$ as

$$S_1(x) = \frac{1}{3}x, \qquad S_2(x) = \frac{1}{3}x + \frac{2}{3}$$

we get F to be the usual Cantor set. The open set condition holds with $J = (0, 1)$, and the decreasing sequence of closed sets F_r above, with $I = \bar{J}$, gives the stages of the famaliar "middle-third" construction.

Even if the open set condition does not hold for any J, there is still a unique nonempty compact F satisfying (2), and this may be constructed from an initial interval I as described. However, the subintervals comprising the F_r will overlap to a considerable degree, and dim F will not necessarily be given by (4). We now show that, nevertheless, the dimensional estimate (4) will "usually" be correct. For example, it follows from Theorem 1 that if

$$S_i(x) = \frac{1}{4}x + c_i \qquad (i = 1, 2, 3)$$

the invariant set F (which may be thought of as a Cantor set of "overlapping" construction, see Fig. 1) will have dimension log 3/log 4 for almost all $(c_1, c_2, c_3) \in \mathbb{R}^3$ in the sense of three-dimensional Lebesgue measure.

To avoid undue technical difficulties, we present the basic theorem in one dimension. The obvious higher dimensional extensions are probably true, and an analogous proof works in many cases.

The idea of the proof is to convert the problem into a related one for a set of similarities on $\mathbb{R}^k$ for which the open set condition *does* hold. The dimension of the corresponding invariant set can be found, and this set is "projected" back to $\mathbb{R}^1$ to give invariant sets for the mappings in the original problem. Use is made of the projection theorems for Hausdorff dimensions (originally due to Marstrand[5]; see also Falconer,[2] Chapter 6) in the following form. If $F \subset \mathbb{R}^k$ is a Borel set, then for almost all

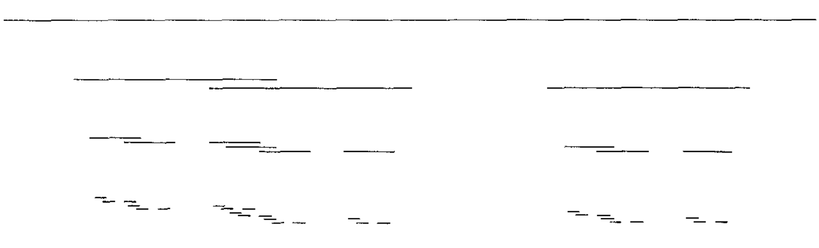


Fig. 1. Construction of an overlapping Cantor set.

$\mathbf{t}=(t_1, t_2,..., t_{k-1})$ in the sense of $(k-1)$-dimensional Lebesgue measure we have $\dim U_{\mathbf{t}}(F)=\min\{1, \dim F\}$, where $U_{\mathbf{t}}: \mathbb{R}^k \to \mathbb{R}$ is given by

$$U_{\mathbf{t}}(x_0, x_1,..., x_{k-1})=x_0+t_1x_1+\cdots+t_{k-1}x_{k-1}.$$

In the statement of the theorem the k degrees of freedom as $(c_1,..., c_k)$ ranges through $\mathbb{R}^k$ are slightly illusory, since the invariant sets are identical to within translations on each of a family of parallel lines in $\mathbb{R}^k$.

We write T_i+c_i to denote the mapping $x \to T_ix+c_i$.

Theorem 1. Let $T_i: \mathbb{R} \to \mathbb{R}(1 \leqslant i \leqslant k)$ be the linear contractions $T_ix=\lambda_ix$, with $0<|\lambda_i|<1$ and $\sum_1^k |\lambda_i|<1$. Then for almost all $(c_1,..., c_k) \in \mathbb{R}^k$ in the sense of k-dimensional Lebesgue measure the nonempty compact set $E \subset \mathbb{R}$ satisfying

$$E=\bigcup_{i=1}^{k} (T_i+c_i)\, E$$

has Hausdorff dimension s, where $\sum_1^k |\lambda_i|^s=1$.

Proof. We lift the mappings T_i to $\mathbb{R}^k$ by letting $S_i: \mathbb{R} \times \mathbb{R}^{k-1} \to \mathbb{R} \times \mathbb{R}^{k-1}$ be given by

$$S_i(x, \mathbf{y})=(\lambda_ix, \lambda_i\mathbf{y}+\mathbf{a}_i)$$

By the conditions on the λ_i, we may choose points $\mathbf{a}_i=(a_{i,1},..., a_{i,k-1}) \in \mathbb{R}^{k-1}$ so that the hypercubes $S_i(J)$ are mutually disjoint and contained in J, where $J=(-1, 1)^k$. Moreover, by making small displacements in the $\mathbf{a}_i$ if necessary, we may also assume that the vectors with components $(a_{i,1},..., a_{i,k-1}, 1-\lambda_i)$ $(1 \leqslant i \leqslant k)$ span $\mathbb{R}^k$. Thus, the mappings S_i are similarity mappings of ratios $|\lambda_i|$ on $\mathbb{R}^k$ with J satisfying the open set condition.

Thus, there is a unique, nonempty compact $F \subset \mathbb{R}^k$ such that, $F=\bigcup_{i=1}^k S_i(F)$. Moreover, $\dim F=s$, where $\sum_1^k |\lambda_i|^s=1$.

For each $\mathbf{t} \in \mathbb{R}^{k-1}$ define a mapping $U_{\mathbf{t}}: \mathbb{R}^k \to \mathbb{R}$ by

$$U_{\mathbf{t}}(x_0, \mathbf{x})=x_0+\mathbf{t}\cdot\mathbf{x}$$

where the dot denotes the scalar product in $\mathbb{R}^{k-1}$. It is trivial to verify that for all $\mathbf{t} \in \mathbb{R}^{k-1}$ and $1 \leqslant i \leqslant k$ the diagram

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "S_i"] \arrow[d, "U_{\mathbf{t}}"'] & \mathbb{R}^k \arrow[d, "U_{\mathbf{t}}"] \\ \mathbb{R}^1 \arrow[r, "T_i+\mathbf{a}_i\cdot\mathbf{t}"'] & \mathbb{R}^1 \end{tikzcd}$$

commutes. Thus,

$$U_{\mathbf{t}}(F)=\bigcup_{i=1}^{k} U_{\mathbf{t}}\circ S_i(F)=\bigcup_{i=1}^{k} (T_i+\mathbf{a}_i\cdot\mathbf{t})(U_{\mathbf{t}}(F))$$

so that $U_{\mathbf{t}}(F)$ is the nonempty, compact, invariant set associated with the mappings $(T_i+\mathbf{a}_i\cdot\mathbf{t})$ $(1\leqslant i\leqslant k)$. By the projection theorems, dim $U_{\mathbf{t}}(F)=s$ for almost all $\mathbf{t}\in\mathbb{R}^{k-1}$. For any t_0, we have

$$(U_{\mathbf{t}}(F)+t_0)=\bigcup_{i=1}^{k} (T_i+\mathbf{a}_i\cdot\mathbf{t}+(1-\lambda_i)\,t_0)(U_{\mathbf{t}}(F)+t_0)$$

Since the dimension of a set is invariant under translation, $\dim(U_{\mathbf{t}}(F)+t_0)=s$ for almost all $(t_0,\mathbf{t})\in\mathbb{R}\times\mathbb{R}^{k-1}$. But the mapping

$$(t_0,\mathbf{t})\to(\mathbf{a}_i\cdot\mathbf{t}+(1-\lambda_1)\,t_0,\ldots,\mathbf{a}\cdot\mathbf{t}+(1-\lambda_k)\,t_0)$$

is a linear bijection on $\mathbb{R}^k$, so that the unique invariant sets for the mappings $\{T_1+c_1,\ldots,T_k+c_k\}$ have dimension s for almost all $(c_1,\ldots,c_k)\in\mathbb{R}^k$. ∎

It is possible to estimate how often the invariant set can have exceptionally small dimension using more delicate versions of the projection theorems. For example, using results of Mattila,[7] Section 5.7(1), it can be shown in exactly the same way that if $u\leqslant s$, dim $E\geqslant u$ for all $(c_1,\ldots,c_k)\in\mathbb{R}^k$ except for a set of zero $(k-1+u)$-dimensional Hausdorff measure.

3. THE DIMENSIONS OF ATTRACTORS OF PIECEWISE LINEAR MAPPINGS

Much has been written on the nature of attractors of injective transformations of plane domains. A particularly fundamental type of attractor results from a horseshoe type of mapping. Some of the principal features of such mappings may be observed in certain piecewise linear transformations, such as the aptly named "baker's transformation" (see Ref. 3 for some versions of this). For transformations that are not injective, considerable problems arise, for example, in connection with the Hausdorff dimension of attractors. It is reasonable to suppose that if the pieces that make up an attractor overlap, the dimansion might be smaller than otherwise would be expected. Again, it is natural to examine piecewise linear transformations, and Alexander and Yorke[1] have shown, in the case of "fat baker's transformations," that this reduction in dimension can occur. However, it is widely believed that such occurrences are exceptional, and we now use a variant of the method employed in Theorem 1 to show that this is indeed the case for a class of piecewise linear transformations, which might be termed "slanting baker's transformations."

Fix $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$, with $|\lambda_1| + |\lambda_2| < 1$. For each $c_1, c_2 \in \mathbb{R}$ define $T_1^{c_1}, T_2^{c_2}: \mathbb{R} \times [-1, 1] \circlearrowleft$ by

$$T_1^{c_1}(x, y) = (\lambda_1 x + \mu_1 y + c_1, 2y - 1)$$

$$T_2^{c_2}(x, y) = (\lambda_2 x + \mu_2 y + c_2, 2y + 1)$$

Let $T^{(c_1, c_2)}: \mathbb{R} \times [-1, 1] \circlearrowleft$ be given by

$$T^{(c_1, c_2)}(x, y) = \begin{cases} T_1^{c_1}(x, y) & \text{if } y \geqslant 0 \\ T_2^{c_2}(x, y) & \text{if } y < 0 \end{cases}$$

It is clear that for each (c_1, c_2), if K is a sufficiently large compact interval, the set $K \times [-1, 1]$ is mapped into itself by $T^{(c_1, c_2)}$. For transformations of this type, it is easy to see that the attractor of $T^{(c_1, c_2)}$ is the ω-limit set obtained as a decreasing sequence of iterates

$$\bigcap_{r=1}^{\infty} (T^{(c_1, c_2)})^{(r)} (K \times [-1, 1])$$

and also that this set is independent of the particular K chosen (see Fig. 2).

Theorem 2. For almost all $(c_1, c_2) \in \mathbb{R}^2$ the attractor of the transformation $T^{(c_1, c_2)}: \mathbb{R} \times [-1, 1] \circlearrowleft$ as above has Hausdorff dimension $1 + s$, where $|\lambda_1|^s + |\lambda_2|^s = 1$.

Proof. Choose a compact interval $I \subset \mathbb{R}$ large enough so that $\lambda_1 I + \mu_1 y \subset I$ and $\lambda_2 I + \mu_2 y \subset I$ for all $-1 \leqslant y \leqslant 1$. Define mappings $S_i: \mathbb{R}^3 \circlearrowleft$ by

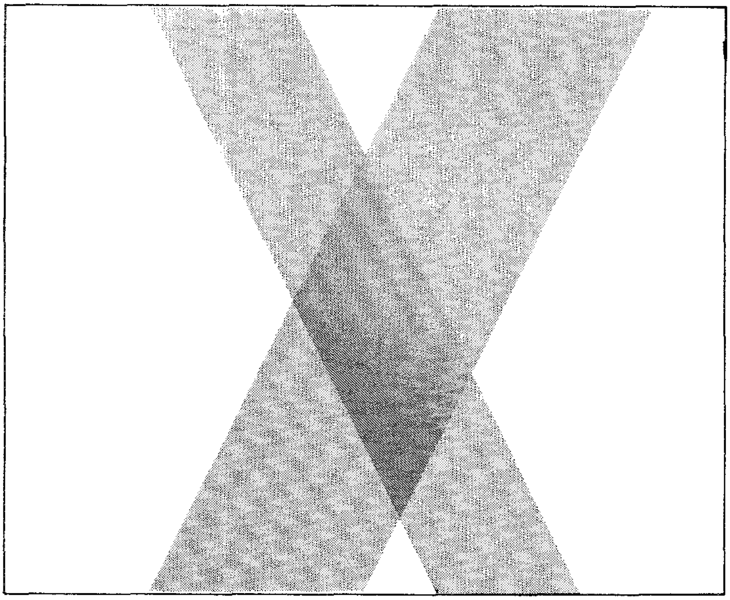
$$S_1(x, y, z) = (\lambda_1 x + \mu_1 y, 2y - 1, \lambda_1 z + a_1)$$

$$S_2(x, y, z) = (\lambda_2 x + \mu_2 y, 2y + 1, \lambda_2 z + a_2)$$

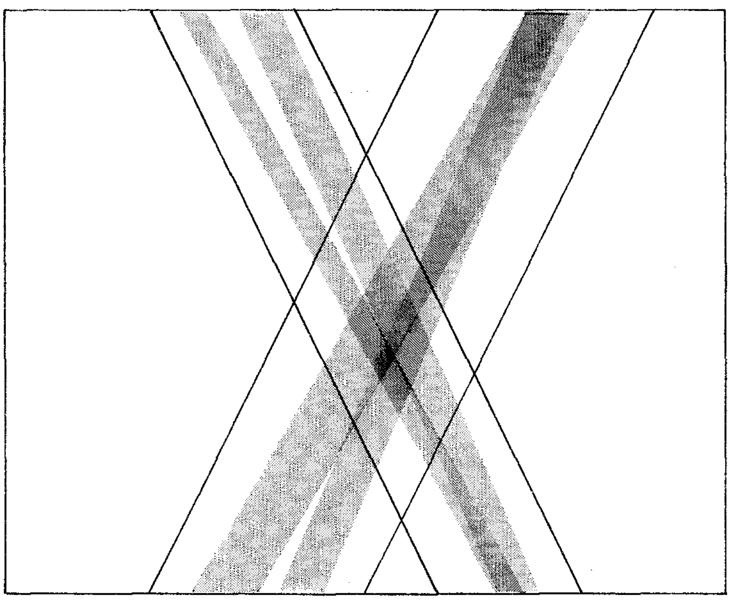
Since $|\lambda_1| + |\lambda_2| < 1$, we may choose number $a_1, a_2 \in \mathbb{R}$ so that $(\lambda_1 I + a_1) \cup (\lambda_2 I + a_2) \subset I$ with the union disjoint and also so that $a_1(1 - \lambda_2) \neq a_2(1 - \lambda_1)$. Write $D = I \times [-1, 1] \times I \subset \mathbb{R}^3$, and define $S: D \to D$ by

$$S(x, y, z) = \begin{cases} S_1(x, y, z), & 0 \leqslant y \leqslant 1 \\ S_2(x, y, z), & -1 \leqslant y < 0 \end{cases}$$

Let P_y denote the plane containing those points in $\mathbb{R}^3$ with second coordinate equal to y. For each r the iterate $S^{(r)}(D)$ consist of 2^r disjoint



(a)



(b)

Fig. 2. First two iterations of the attractor of a slanting baker's transformation.

parallelpipeds between the planes P_{-1} and P_1. These parallelepipeds are nested in the sense that each parallelepiped of $S^{(r)}(D)$ contains two of those of $S^{(r+1)}(D)$.

Similarly, for each $-1 \leqslant y \leqslant 1$, $P_y \cap S^{(r)}(D)$ consists of 2^r disjoint squares of side lengths $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}$ $(i_j = 1, 2)$ times those of $I \times I$. As r increases, these squares are nested in the natural way. Let $F = \bigcap_{r=1}^{\infty} S^{(r)}(D)$. Standard methods (compare Falconer,[2] Theorem 8.6) show that the set $P_y \cap F = \bigcap_{r=1}^{\infty} (P_y \cap S^{(r)}(D))$ has Hausdorff dimension s for all $-1 \leqslant y \leqslant 1$, where $|\lambda_1|^s + |\lambda_2|^s = 1$.

Let $U_t(x, y, z) = (x + tz, y)$ for $t \in \mathbb{R}$. By considering the cases $y < 0$, $y \geqslant 0$ independently, it is easy to verify that the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "S"] \arrow[d, "U_t"'] & D \arrow[d, "U_t"] \\ \mathbb{R} \times [-1, 1] \arrow[r, "T^{(a_1 t, a_2 t)}"'] & \mathbb{R} \times [-1, 1] \end{tikzcd}$$

For each t and r, we have

$$(T^{(a_1 t, a_2 t)})^{(r)} \, (U_t(D)) = U_t(S^{(r)}(D))$$

so, taking the limits of this decreasing sequence of compact sets,

$$\bigcap_{r=1}^{\infty} \, (T^{(a_1 t, a_2 t)})^{(r)} \, (U_t(D)) = U_t(F)$$

Thus, for each value of t, $U_t(F)$ is the ω-limit set, and thus the attractor, of $T^{(a_1 t, a_2 t)}$.

From the definition of U_t, we have $P_y \cap U_t(F) = U_t(P_y \cap F)$. For all y, $\dim(P_y \cap F) = s$, so it follows from the projection theorems that

$$s = \dim U_t(P_y \cap F) = \dim(P_y \cap U_t(F))$$

for almost all t. Routine arguments give that $(y, t) \to \dim U_t(P_y \cap F)$ is a Borel function, so it follows by Fubini's theorem that for almost all $t \in \mathbb{R}$, we have $\dim(P_y \cap U_t(F)) = s$ for almost all $-1 \leqslant y \leqslant 1$. By the result of Marstand[6] (or see Falconer,[2] Theorem 5.8), $\dim U_t(F) \geqslant 1 + s$ for almost all t. On the other hand, for each r the set $U_t(F)$ is contained in $(T^{(a_1 t, a_2 t)})^{(r)} \, (D)$, a set formed by 2^r (overlapping) parallelograms of widths $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_r}$ with $\sum_{i,j=1,2} |\lambda_{i_j}|^s = 1$. A direct covering argument gives $\dim U_t(F) \leqslant 1 + s$ for all t. We conclude that the attractor $U_t(F)$ of $T^{(a_1 t, a_2 t)}$ has dimension s, for almost all t.

Finally note that for $t_0 \in \mathbb{R}$

$$(t_0, 0) + T^{(c_1,c_2)}(x - t_0, y) = T^{(c_1 + (1-\lambda_1)t_0, c_2 + (1-\lambda_2)t_0)}(x, y)$$

Thus, the attractors of

$$T^{(a_1 t, a_2 t)} \qquad \text{and} \qquad T^{(a_1 t + (1-\lambda_1)t_0, a_2 t + (1-\lambda_2)t_0)}$$

are translates of each other and therefore have the same dimension. Letting (t, t_0) run through $\mathbb{R}^2$, the result follows. ∎

Again, by using more delicate projection theorems, one can show that the attractor has dimension at least $u \leqslant s$ except for a set of $(c_1, c_2) \in \mathbb{R}^2$ of zero $(1+u)$-dimensional Hausdorff measure.

It is easy to see that the Liapunov dimension of the transformations $T^{(c_1,c_2)}$ considered in Theorem 2 is $1 - (2 \log 2)/\log |\lambda_1 \lambda_2|$. This equals the "almost sure" Hausdorff dimension if $\lambda_1 = \lambda_2$.

4. CONCLUSION

The method described here certainly has much wider applications to sets invariant under a collection of linear mappings and to attractors of piecewise linear transformations. In the form described here, the method is essentially linear. However, there are good reasons for expecting similar results to hold for invariant sets and attractors in the nonlinear case. For example, if T^c, $c \in D$, is a sufficiently large family of (not necessarily injective) transformations of a plane region, then the dimension of the attractor should be an essentially continuous function of c, so that the restriction of T^c to $D - N$ is continuous in c, where N has, in some sense, measure zero. It is hoped to incorporate these ideas in a future paper.

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